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Alternating Trigonometric Polynomials

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Trigonometric polynomials induced by equioscillation with respect to a given periodic function $f(t)$ at appropriately shifted equally spaced nodes are introduced. Two sequences of functionals $A_n(f)$, $B_n(f)$ $(n = 1, 2,...)$, corresponding to the specific choice of shift-parameters are defined and their approximation properties are investigated. It is shown that these functionals are closely related to the Fourier coefficients of $f(t)$. It is proved that under some conditions an approximated function is determined by $A_n(f)$, $B_n(f)$, $n = 1, 2,...$, uniquely up to an additive constant. It is also shown that the rate at which $A_n(f)$ and $B_n(f)$ approach zero gives valuable information about the differential properties of $f(t)$. © 1987 Academic Press, Inc.

I. INTRODUCTION

Let $C[-1, 1]$ be the Banach space of real continuous functions on $[-1, 1]$ equipped with the sup norm $(\|f\| = \max_{-1 \leqslant x \leqslant 1} |f(x)|)$ and denote by π_n the subspace of $C[-1, 1]$ consisting of all real algebraic polynomials of degree $\leqslant n$. To every $f(x) \in C[-1, 1]$ there corresponds the unique polynomial of best uniform approximation $e_n(f; x)$ satisfying $E_n(f) = \|f - e_n\| = \min_{p \in \pi_n} \|f - p\|$. The well-known Chebyshev characterization theorem establishes a close relation between the optimality of $e_n(f; x)$ and the existence of a special set of points $\{x_k^*\}_{k=0}^{n+1}$, $-1 \leqslant x_{n+1}^* < x_n^* < \cdots < x_1^* < x_0^* \leqslant 1$ (a so-called Chebyshev alternation set) such that

$$f(x_k^*) - e_n(f; x_k^*) = (-1)^k \gamma E_n(f), \qquad k = 0, 1,..., n+1, \tag{1}$$

where γ equals 1 or -1.

Since one has usually no prior knowledge about the location of such points x_k^*, indirect methods were proposed for approximate determination of $e_n(f; x)$. The first step in this direction was taken by de La Vallée-

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Poussin [6], who introduced the polynomials $v_n(f; X; x)$ defined together with the parameters $V_n(f; X)$ by the linear system

$$f(x_k) - v_n(f; X; x_k) = (-1)^k V_n(f; X), \qquad k = 0, 1, ..., n+1, \tag{2}$$

where $X = \{x_k\}_{k=0}^{n+1}$ is a fixed set of $n+2$ (distinct) points of $[-1, 1]$. We will refer to the v_n-polynomials as the alternating polynomials. The functional $V_n(f; X)$ is known to be of great theoretical importance, since [7]

$$|V_n(f; X)| \leqslant E_n(f). \tag{3}$$

The investigations of de La Vallée-Poussin have been continued by Bernstein [1], who posed the important question of a favorable choice of the set X and showed that for a family of functions analytic in a complex domain containing the interval $[-1, 1]$ the points $\{x_k^*\}_{k=0}^{n+1}$ of the Chebyshev alternation set tend asymptotically to the fixed set of points $\hat{T} = \{\cos[k\pi/(n+1)]\}_{k=0}^{n+1}$. Under the assumption that $f(x)$ may be expanded in an absolutely convergent Fourier–Chebyshev series $f(x) = c_0/2 + \sum_{k=1}^{\infty} c_k T_k(x)$, Bernstein also established the relationship [2]

$$V_n(f; \hat{T}) = c_{n+1} + c_{3(n+1)} + c_{5(n+1)} + \cdots. \tag{4}$$

Motivated by the results of Bernstein, Eterman initiated the investigation of approximation properties of the $v_n(f; \hat{T}; x)$-polynomials [8, 10]. Some representations of the $v_n(f; \hat{T}; x)$-polynomials and the $V_n(f; \hat{T})$-functionals may also be found in Meinardus [16], Rivlin [19], and Phillips and Taylor [18]. The operator norm of $v_n(f; \hat{T}; x)$ was estimated by Malozemov [14] and Cheney and Rivlin [5]. Several numerical applications of the $v_n(f; \hat{T}; x)$-polynomials have been discussed by Eterman [9, 11] and Brutman [3, 4]. It should be pointed out that the effectiveness of the $v_n(f; \hat{T}; x)$-polynomials as an approximation tool is based on the fact that for sufficiently smooth functions, $v_n(f; \hat{T}; x)$ is a good approximation to the minimax polynomial while the parameter $V_n(f; \hat{T})$ may serve as a reasonable estimator of the approximation error.

Turning now to the periodic case we first mention the well-known fact that by analogy with (1) for any $f(t)$ belonging to the space $C_{2\pi}$ of continuous 2π-periodic functions, there exists the unique minimax trigonometric polynomial $\tilde{e}_n(f; t)$ satisfying

$$f(t_k^*) - \tilde{e}_n(f; t_k^*) = (-1)^k \gamma \tilde{E}_n(f), \qquad k = 0, 1, ..., 2n+1, \tag{5}$$

where $0 \leqslant t_0^* < t_1^* < \cdots < t_{2n} < 2\pi$, γ equals 1 or -1, and $\tilde{E}_n(f) = \|f - \tilde{e}_n(f)\|$. Thus in the trigonometric case the Chebyshev alternation set $\{t_k^*\}_{k=0}^{2n+1}$ consists of (at least) $2n+2$ distinct points in $[0, 2\pi)$. Taking into account this analogy and keeping in mind the above-mentioned valuable approximation properties of the $v_n(f; \hat{T}; x)$-polynomials, one can pose a natural question of how to extend the notion of the $v_n(f; \hat{T}, x)$-polynomials to the trigonometric approximation.

The first, and to our knowledge the single, attempt to construct a trigonometric analog of the $v_n(f; \hat{T}; x)$-polynomials is due to Malozemov [15]. In this paper the author introduced the trigonometric polynomials of degree $\leqslant n$ which alternate with respect to a given 2π-periodic function at the set of points $\{k\pi/(n+1)\}_{k=0}^{2n+1}$. Malozemov established some properties of these polynomials and estimated the norm of the corresponding operator. Cheney and Rivlin obtained a precise expression for the operator norm [5]. It should be emphasized that the approach of Malozemov does not take into considertion the inherent difference that exists between the algebraic and the trigonometric approximation. In the algebraic case the choice of the $\hat{T}$-nodes as an alternation set is justified by Bernstein's result which guarantees that for sufficiently smooth functions the Chebyshev alternant tends asymptotically to the $\hat{T}$-set of points. In particular, for algebraic polynomials of degree $(n+1)$ the Chebyshev alternation set coincides with the $\hat{T}$-set for every n. In contrast, for a trigonometric polynomial of degree $(n+1)$ $\alpha\cos(n+1)t+\beta\sin(n+1)t$, the Chebyshev alternation set, which is known to be $\{k\pi/(n+1)+\arctan(\beta/\alpha)\}_{k=0}^{2n+1}$ (see [17]), depends on the ratio β/α. In other words, in the periodic case the location of the Chebyshev alternant depends also on the "degree of evenness" of $f(t)$.

Motivated by this observation we introduce in the present paper the trigonometric polynomials induced by equioscillation at appropriately shifted equally spaced nodes. Our attention is concentrated on two specific choices of the shift-parameter by means of which two sequences of functionals $A_n(f)$, $B_n(f)$, $n=1, 2,...$, are defined. The study of the characteristic properties of these functionals, which Section 3 is devoted to, is a main purpose of this paper. We first prove (Theorem 2) that under some slight restrictions on the order of the Fourier coefficients an approximated function $f(t)$ is determined by the sequences $A_n(f)$, $B_n(f)$, $n=1, 2, 3,...$, uniquely up to an additive constant. Theorem 3, together with the representations (12), (13), establishes a close relationship between the Fourier coefficients and the functionals $A_n(f)$, $B_n(f)$.

In conclusion we study the relationship between the differential properties of $f(t)$ and the rate at which $A_n(f)$ and $B_n(f)$ approach zero. We prove for illustration one result of this type (Theorem 3), namely, that if $A_n(f)=O(1/n^{1+\alpha})$, $B_n(f)=O(1/n^{1+\alpha})$, then $f(t)$ belongs to the class Lip α. This result is similar to the result of Lorentz concerning the relationship between the differential properties of an approximated function and the behavior of its Fourier coefficients [13].

II. Alternating Trigonometric Polynomials

Let $f(t) \in C_{2\pi}$ and $t_k = h_n + \{k\pi/(n+1)\}$, $k = 0, 1, \ldots, 2n+1$, $0 \leqslant h_n < \pi/(n+1)$. We introduce the trigonometric polynomial $v_n(f; h_n; t) = \alpha_0/2 + \sum_{v=1}^{n} (\alpha_v \cos vt + \beta_v \sin vt)$, the coefficients of which are defined together with the parameter $V_n(f; h_n)$ from the linear system

$$f(t_k) - v_n(f; h_n, t_k) = (-1)^k V_n(f; h_n), \qquad k = 0, 1, \ldots, (2n+1). \tag{6}$$

Theorem 1. *For any $f(t) \in C_{2\pi}$ and any $h_n (0 \leqslant h_n < \pi/(n+1))$, there exists a unique trigonometric polynomial $v_n(f; h_n; t)$.*

Proof. It follows from (6) that

$$\sum_{k=0}^{2n+1} (-1)^k \{(-1)^k V_n(f; h_n) + v_n(f; h_n; t_k)\} = \sum_{k=0}^{2n+1} (-1)^k f(t_k). \tag{7}$$

Taking into account that

$$\begin{aligned} \sum_{k=0}^{2n+1} (-1)^k e^{imt_k} &= e^{imh_n}(2n+2), \qquad m = (2\lambda+1)(n+1), \quad \lambda = 0, 1, 2, \ldots, \\ &= 0, \qquad \text{otherwise,} \end{aligned} \tag{8}$$

we find

$$V_n(f; h_n) = \frac{1}{2n+2} \sum_{k=0}^{2n+1} (-1)^k f\left(k\frac{\pi}{n+1} + h_n\right). \tag{9}$$

After the substitution of (9) in (7), we arrive at a system of $(2n+1)$ equations for the coefficients $\alpha_0, \alpha_1, \beta_1, \ldots, \alpha_n, \beta_n$. As may be easily verified the determinant of this system for any h_n $(0 \leqslant h_n < \pi/(n+1))$ does not vanish and thus the theorem follows.

By analogy with the algebraic case, the polynomials $v_n(f; h_n; t)$ will be called the alternating trigonometric polynomials. For fixed h_n the polynomial $v_n(f; h_n; t)$ may be interpreted as a linear projection from $C_{2\pi}$ onto the trigonometric polynomials of degree $\leqslant n$, while the parameter $V_n(f; h_n)$ is a linear functional on $C_{2\pi}$. The inequality of de La Vallée-Poussin (see e.g., [7]) involves

$$\max_{0 \leqslant h_n < \pi/(n+1)} |V_n(f; h_n)| \leqslant \tilde{E}_n(f). \tag{10}$$

Note that the equality in (10) holds if, in particular, $f(t)$ is a trigonometric polynomial of degree $(n+1)$.

Turning to the question of a favorable choice of the shift-parameter h_n, let us suppose that $f(t)$ belongs to the class F of functions which can be expanded in an absolutely convergent Fourier series:

$$F \stackrel{\text{def}}{=} \left\{ f(t) = a_0/2 + \sum_{k=1}^{\infty} [a_k \cos kt + b_k \sin kt],\ \sum_{k=1}^{\infty} (|a_k| + |b_k|) < \text{const} \right\}.$$

Then it follows from (9), by taking (8) into account, that

$$V_n(f; h_n) = \sum_{s=0}^{\infty} \{ a_{(2s+1)(n+1)} \cdot \cos[(2s+1)(n+1)\, h_n] + b_{(2s+1)(n+1)} \cdot \sin[(2s+1)(n+1)\, h_n] \}. \tag{11}$$

In particular, if we define $A_n(f) = V_n(f; 0)$ and $B_n(f) = V_n(f; \pi/2(n+1))$, then

$$A_n(f) = a_{n+1} + a_{3(n+1)} + a_{5(n+1)} + \cdots, \qquad n = 0, 1, ..., \tag{12}$$

$$B_n(f) = b_{n+1} - b_{3(n+1)} + b_{(5n+1)} - \cdots, \qquad n = 0, 1, ... \tag{13}$$

Note that (12) is identical with Bernstein's representation (4) for the algebraic case. The lower bound for $\tilde{E}_n(f)$ in terms of $A_n(f)$, $B_n(f)$ will be

$$\{A_n^2(f) + B_n^2(f)\}^{1/2} \leqslant \sqrt{2}\, \tilde{E}_n(f). \tag{14}$$

Since $A_n(f) = 0$, $n = 0, 1, ...$, when $f(t)$ is odd, while $B_n(f) = 0$, $n = 0, 1, ...$, for even $f(t)$, the ratios $B_n(f)/A_n(f)$ measure the "degree of evenness" of $f(t)$ and may be used to determine a suitable value for the shifrt-parameter h_n. For sufficiently smooth functions it is reasonable to accept that a favorable value h_n^* corresponds to the requirement of maximizing the principal part of expansion (11). This together with (12), (13) leads to the approximate formula

$$h_n^* \approx \frac{1}{n+1} \arctan \frac{B_n(f)}{A_n(f)}. \tag{15}$$

We conclude this section by noting that the alternating trigonometric polynomials based on the equally spaced nodes with the shift-parameter defined by (15) may be recommended as a first approximation to the minimax polynomial in constructng the trigonometric version of the Remes algorithm.

III. Properties of the Functionals $A_n(f)$, $B_n(f)$

It has been shown by Eterman [10] that in the algebraic case for a wide class of functions the sequence $V_n(f; \hat{T})$, $n = 0, 1, \ldots,$ determines an approximated function $f(x)$ uniquely up to an additive constant. In the following we prove an analogous characteristic property of the sequences $A_n(f)$, $B_n(f)$ for the trigonometric case. We start with the following lemma.

LEMMA 1. *The two linear systems*

$$\sum_{v=0}^{\infty} c_{(2v+1)n} = 0, \qquad n = 1, 2, 3, \ldots, \tag{16}$$

$$\sum_{v=0}^{\infty} (-1)^v c_{(2v+1)n} = 0, \qquad n = 1, 2, 3, \ldots, \tag{17}$$

with $\sum_{n=1}^{\infty} |c_n| < \infty$, *admit only the trivial solution.*

Proof. Since the proof of the lemma for the first system is given in [10], we restrict ourselves to (17) and apply the method of [10]. Let $S_n = \sum_{v=0}^{\infty} (-1)^v c_{(2v+1)n}$ and denote by $p_1, p_2, \ldots, p_k, \ldots$ the successive primes starting with $p_1 = 3$. First, we observe that all $c_{(2s+1)np_i}$ appear in S_{np_i} either with the same signs as in S_n or with the opposite signs. Let ε_{np_i} denote the sign with which c_{np_i} appears in S_n. In order to prove that $c_n = 0$ consider the sum

$$S_n - (\varepsilon_{np_1} S_{np_1} + \cdots + \varepsilon_{np_k} S_{np_k}) + (\varepsilon_{np_1p_2} S_{np_1p_2} + \cdots + \varepsilon_{np_{k-1}p_k} S_{np_{k-1}p_k})$$
$$- \cdots + (-1)^k \varepsilon_{np_1p_2\cdots p_k} S_{np_1p_2\cdots p_k}. \tag{18}$$

We claim that this sum contains no elements C_{nl}(l-odd) with $1 < l < p_{k+1}$, while c_n and the elements c_{nl} corresponding to $l \geqslant p_{k+1}$ appear in this sum at most once. Indeed, the assertion is obvious in case the canonical prime number decomposition of l does not contain $p_1, p_2, \ldots, p_k$. Otherwise (i.e., if $l = p_{i_1}^{\alpha_1} \cdots p_{i_m}^{\alpha_m}$ with $p_1 \leqslant i_1 < \cdots < i_m \leqslant p_k$), c_{nl} appears once in S_n; $\binom{m}{1}$ times in the first group $\{S_{np_1}, \ldots, S_{np_k}\}$, $\binom{m}{2}$ times in the second group $\{S_{np_1p_2}, \ldots, S_{np_{k-1}p_k}\}$, and so on. Taking into account the signs, we find that in this case c_{nl} appears

$$1 - \binom{m}{1} + \binom{m}{2} - \cdots + (-1)^m \binom{m}{m} = 0 \text{ times}$$

Hence

$$c_n = -\varepsilon_{np_{k+1}} c_{np_{k+1}} - \cdots,$$

and

$$|c_n| \leqslant \sum_{r=np_k+1}^{\infty} |c_r|. \tag{19}$$

Letting now $k \to \infty$, we get $c_n = 0$, $n = 1, 2, 3, \ldots$.

Now we are in a position to prove the following.

THEOREM 2. *Let $f_1(t), f_2(t) \in F$, and for any $n = 0, 1, 2, \ldots$,*

$$\begin{aligned} A_n(f_1) &= A_n(f_2), \\ B_n(f_1) &= B_n(f_2). \end{aligned} \tag{20}$$

Then

$$f_1(t) = f_2(t) + \text{const.} \tag{21}$$

Proof. Let $f(t) = f_1(t) - f_2(t)$. Then $f(t) \in F$ and hence $f(t) = a_0/2 + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt)$ with $\sum_{n=1}^{\infty} |a_n| < \infty$ and $\sum_{n=1}^{\infty} |b_n| < \infty$. Furthermore, relations (20) yield that $A_n(f) = B_n(f) = 0$, $n = 0, \ldots$. This implies in view of (12), (13) that

$$\sum_{s=0}^{\infty} a_{(2s+1)n} = 0, \qquad n = 1, 2, \ldots,$$

$$\sum_{s=0}^{\infty} (-1)^s b_{(2s+1)n} = 0, \qquad n = 1, 2, \ldots.$$

It remains to apply Lemma 1; the result follows.

For the derivation of our next result we need some properties of the number-theoretic Möbius function $\mu(n)$, which is defined as [12]

$$\mu(n) = \begin{cases} 1, & \text{if } n = 1, \\ (-1)^s, & \text{if } n \text{ is the product of } s (s \geqslant 1) \text{ distinct primes,} \\ 0, & \text{otherwise, i.e., if the square of at least one prime divides } n. \end{cases} \tag{22}$$

The following is a basic property of $\mu(n)$ [12]:

$$\sum_{d/n} \mu(d) = \begin{cases} 1 & \text{for} \quad n = 1, \\ 0 & \text{for} \quad n > 1, \end{cases} \tag{23}$$

where the summation is performed over all the divisors d of the number n.

In order to give an effective algorithm for recovering $f(t)$ by means of the sequences $A_n(f)$, $B_n(f)$, $n=0, 1, 2,...$, we introduce the class F_α of functions such that the Fourier coefficients are of the order

$$a_n = 0\left(\frac{1}{n^{1+\alpha}}\right), \qquad b_n = 0\left(\frac{1}{n^{1+\alpha}}\right), \qquad \alpha > 0. \tag{24}$$

THEOREM 3. *Let $A_n(f)$, $B_n(f)$, $n=0, 1,...$, for $f \in F_\alpha$ be given. Then there exists a unique solution to each of the following two linear systems for the Fourier coefficients of $f(t)$:*

$$A_{n-1} \equiv A_{n-1}(f) = \sum_{m=0}^{\infty} a_{(2m+1)n}, \qquad n = 1, 2, 3,..., \tag{25}$$

$$B_{n-1} \equiv B_{n-1}(f) = \sum_{m=0}^{\infty} (-1)^m b_{(2m+1)n}, \qquad n = 1, 2, 3,.... \tag{26}$$

The solutions are given by

$$a_n = \sum_{s=0}^{\infty} \mu(2s+1)\, A_{(2s+1)n-1}, \qquad n = 1, 2,..., \tag{27}$$

$$b_n = \sum_{s=0}^{\infty} (-1)^s \mu(2s+1)\, B_{(2s+1)n-1}, \qquad n = 1, 2,.... \tag{28}$$

Proof. Since (27) has been established in [10], we restrict ourselves to proving (28). Conditions (24) guarantee, in view of Lemma 1, that the system (26) has a unique solution. To show that this solution is of the form (28), note first that for any $f(t) \in F_\alpha$ there is absolute convergence in (28), since after the summation of the two sides of (13), we find

$$\sum_{n=0}^{\infty} |B_n| < \sum_{n=1}^{\infty} \tau(n)\, |b_n|. \tag{29}$$

In (22), $\tau(n)$ denotes the number of divisors of n, which is known to satisfy [12]

$$\lim_{n \to \infty} \frac{\tau(n)}{n^\varepsilon} = 0, \qquad \text{any} \quad \varepsilon > 0. \tag{30}$$

By substituting on the right-hand side of (13) in place of b_n the expression (28), we get

$$\sum_{m=0}^{\infty} (-1)^m b_{(2m+1)n} = \sum_{m=0}^{\infty} \sum_{s=0}^{\infty} (-1)^{m+s} \mu(2s+1)\, B_{(2s+1)(2m+1)n-1}. \tag{31}$$

In the double sum above, each $B_{(2\lambda+1)n-1}$ appears a finite number of times, i.e., the number of representations: $(2\lambda+1)=(2s+1)(2m+1)$. It is clear that $(-1)^{m+s}=(-1)^{\lambda}$ and hence

$$\sum_{m=0}^{\infty}\sum_{s=0}^{\infty}(-1)^{m+s}\mu(2s+1)\,B_{(2s+1)(2m+1)n-1}$$
$$=\sum_{\lambda=0}^{\infty}(-1)^{\lambda}B_{(2\lambda+1)n-1}\left\{\sum_{d/2\lambda+1}\mu(d)\right\}. \tag{32}$$

It remains to apply the basic property of the Möbius function (23) which gives

$$\sum_{m=0}^{\infty}\sum_{s=0}^{\infty}(-1)^{m+s}\mu(2s+1)\,B_{(2s+1)(2m+1)n-1}=B_{n-1}.$$

The result follows.

For any $f(t)\in C_{2\pi}$, $A_n(f)$ and $B_n(f)$ tend to zero with $n\to\infty$ at a rate which depends upon the differential properties of $f(t)$. Since $|A_n(f|\leqslant \tilde{E}_n(f)$; $|B_n(f|\leqslant \tilde{E}_n(f)$, one can derive estimates for the order of growth of $A_n(f)$, $B_n(f)$, from the corresponding Jackson results [17]. The more interesting results are those of the converse type, i.e., that the behavior of $A_n(f)$, $B_n(f)$ gives valuable information concerning the differential properties of the aproximated function. We prove here only one result of this kind. Recall that $f(t)\in C_{2\pi}$ is said to belong to the class Lip α if for all t_1, t_2 $|f(t_1)-f(t_2)|\leqslant M\,|t_1-t_2|^{\alpha}$, with some constant M.

THEOREM 4. *Let $f(t)\in F$, and for $0<\alpha<1$, $n=1, 2,...,$*

$$|A_n(f)|\leqslant\frac{\text{const}}{n^{1+\alpha}},$$
$$|B_n(f)|\leqslant\frac{\text{const}}{n^{1+\alpha}}. \tag{33}$$

Then $f(t)$ belongs to the class Lip α.

Proof. Conditions (33) guarantee an absolute convergence of $\sum_{s=0}^{\infty}\mu(2s+1)\,A_{(2s+1)n-1}$ and $\sum_{s=0}^{\infty}(-1)^s\mu(2s+1)\,B_{(2s+1)n-1}$, and hence, by repeating the arguments used in the proof of Theorem 3, we obtain

$$a_n=\sum_{s=0}^{\infty}\mu(2s+1)\,A_{(2s+1)n-1},$$
$$b_n=\sum_{s=0}^{\infty}(-1)^s\mu(2s+1)\,B_{(2s+1)n-1}.$$

Thus for $n \geqslant 2$,

$$|a_n| \leqslant \sum_{s=0}^{\infty} |A_{(2s+1)n-1}| \leqslant \text{const} \sum_{s=0}^{\infty} \frac{1}{[(2s+1)n-1]^{1+\alpha}},$$

$$|b_n| \leqslant \sum_{s=0}^{\infty} |B_{(2s+1)n-1}| \leqslant \text{const} \sum_{s=0}^{\infty} \frac{1}{[(2s+1)n-1]^{1+\alpha}},$$

and hence

$$\begin{aligned} |a_n| &< \frac{\text{const}}{n^{1+\alpha}}, \\ |b_n| &< \frac{\text{const}}{n^{1+\alpha}}. \end{aligned} \tag{34}$$

It remains to apply the result of Lorentz [13] asserting that (34) implies $f(t) \in \text{Lip}\,\alpha$ $(0 < \alpha < 1)$.

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